Solutions to HW # 3

1. Let d be a metric on M. Then $d(x,z) \le d(x,y) + d(y,z)$ by the triangle inequality (property iv). Equivalently,

$d(x,z) - d(y,z) \le d(x,y) \quad (1)$

Similarly, $d(y,z) \le d(y,x) + d(z,x) = d(x,y) + d(x,z)$.

Thus, $d(y,z) - d(x,z) \le d(x,y)$ or

$-\big(d(x,z) - d(y,z)\big) \le d(x,y) \quad (2)$

Now (1) and (2) imply that

$$|d(x,z) - d(y,z)| \le d(x,y) \quad \text{(Why?)}$$

2. First, we show that $d(x,y) = d(y,x)$ by using properties (a) and (b). Setting $z = x$ in (b) and observing that $d(x,x) = 0$, we get

$$d(x,y) \le d(x,x) + d(y,x) = d(y,x). \quad (1)$$

Setting $z = y$ in (b) and observing that, $d(y,y) = 0$, we get

$$d(y,x) \le d(y,y) + d(x,y) = d(x,y). \quad (1)$$

Thus, (1) and (2) imply that

$$d(y,x) \le d(x,y) \le d(y,x).$$

Hence $d(x,y) = d(y,x)$.

To show that $d(x,y) \geq 0$, simply refer to the proof of the previous problem. Namely, any function with the properties

$$(i)\ d(x,y) = d(y,x) \text{ for all } x,y \in M$$

$$(ii)\ d(x,y) \leq d(x,z) + d(y,z) \text{ for all } x,y,z \in M$$

satisfies $0 \leq |d(x,z) - d(y,z)| \leq d(x,y)$.

3. Let $x,y \in M$. Then by properties (i), (ii), (iii) and reverse triangle inequality (rti)

$$0 \underset{(i)}{\overset{\downarrow}{=}} d(x,x) \underset{(rti)}{\overset{\downarrow}{\geq}} d(x,y) + d(y,x) \underset{(iii)}{\overset{\downarrow}{=}} d(x,y) + d(x,y) = 2d(x,y)$$

Thus, $0 \geq d(x,y)$ and since $d(x,y) \geq 0$ by (i), it follows that $0 = d(x,y)$ and, by (ii), $x = y$. Thus M has at most one point, as desired.

4. Let d be a metric on M and suppose $p: M \times M \mapsto [0,\infty)$ is defined by $p(x,y) = \min\{d(x,y), 1\}$. We will prove that p is a metric on M by showing that p satisfies properties (i) – (iv). Properties (i) – (iii) are obvious. To prove (iv), note that $\min\{d(x,y), 1\} \leq 1$ so, for any $a \geq 0$,

$\min\{d(x,y), 1\} \leq a + 1$. (1)

Now $\min\{d(x,y), 1\} \leq d(x,y) \leq d(x,z) + d(z,y)$.

By (1), we see that

$$\min\{d(x,y), 1\} \leq 1 + d(z,y)$$

$$\min\{d(x,y),1\} \leq d(x,z) + 1$$

$$\min\{d(x,y),1\} \leq 1 + 1$$

Hence $\rho(x,y) \leq \min\{d(x,z),1\} + \min\{d(y,z),1\} =$
$= \rho(x,z) + \rho(y,z)$.

5. Let $d = d_1 + d_2$, where d_1 and d_2 are metric functions on M. Then, for $i = 1$ or 2

(i) $0 \leq d_i(x,y) \leq d_1(x,y) + d_2(x,y) < \infty$ for all pairs $x, y \in M$.

(ii) $d_1(x,y) + d_2(x,y) = 0$ if and only if $d_1(x,y) = 0$ and $d_2(x,y) = 0$ if and only if $x = y$.

(iii) $d_1(x,y) + d_2(x,y) = d_1(y,x) + d_2(y,x)$ for all pairs $x, y \in M$.

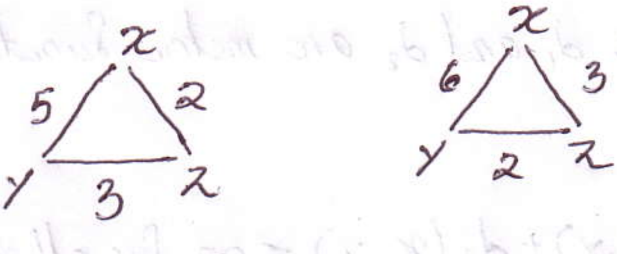
(iv) $d(x,y) = d_1(x,y) + d_2(x,y) \leq d_1(x,z) + d_1(y,z) +$
$+ d_2(x,z) + d_2(y,z) = \left(d_1(x,z) + d_2(x,z)\right) + \left(d_2(y,z) + d_2(y,z)\right)$
$= d(x,z) + d(y,z)$

Thus, $d = d_1 + d_2$ is another metric on M.

Now set $d = \max\{d_1, d_2\}$. Clearly d satisfies properties (i)-(iii). To see that d satisfies property (iv), notice that $d(x,y) = d_i(x,y) \leq d_i(x,z) + d_i(y,z) \leq$
$\leq \max\limits_{i\in\{1,2\}}\{d_i(x,z)\} + \max\limits_{i\in\{1,2\}}\{d_i(y,z)\} = d(x,z) + d(y,z)$

Setting $d = \min\{d_1, d_2\}$, we may find that this time it is not a metric function. Obviously d still satisfies properties (i)-(iii), but what about (iv)?

Suppose $M = \{x, y, z\}$ and d_1 and d_2 are given in the diagram below.



Then $d(x,y) = 5$, while $d(x,z) + d(y,z) = 2 + 2 = 4 < 5$. Thus, the function $\min\{d_1, d_2\}$ does not always satisfy the triangle inequality.

Finally, we will show that d^2 is not generally a metric function. We might as well do this in more generality. Let $1 < \alpha$, we show that if $d = |\cdot|$, then d^α is not a metric on $\mathbb{R}$, because it fails to satisfy property (iv). Notice that

$$|(0.5)^{1/\alpha} - 0| = \left[(0.5)^{1/\alpha}\right]^\alpha = 0.5, \text{ while}$$

$$|(0.5)^{1/\alpha} - \tfrac{1}{2}(0.5)^{1/\alpha}| + |\tfrac{1}{2}(0.5)^{1/\alpha} - 0| = \left(\tfrac{1}{2}\right)^\alpha 0.5 + \left(\tfrac{1}{2}\right)^\alpha 0.5 =$$

$$= \frac{1}{2^\alpha} < \frac{1}{2} = 0.5.$$

6. (a), (c), (d), and (e) are metrics, while (b) is not. Please notice that (b) is of the form $|x-y|^\alpha$, $1 < \alpha$.

Observe that (e) is a sum of two metric functions.

7. If $0<\alpha<1$, then the function $f:[0,\infty)\to[0,\infty)$ given by $f(t)=t^{\alpha}$ is zero iff $t=0$. Notice that $f'(t)>0$ for all $t>0$ and $f''(t)<0$. Thus $f(t+s)\leq$ $\leq f(t)+f(s)$ and for any metric d on M, $f(d)$ is also a metric on M.

Define $p:\mathbb{R}^2\to[0,\infty)$ by $p(x,y)=|x-y|^{\alpha}$, then p is a metric function on $\mathbb{R}$. By exercise 1,

$|p(x,0)-p(y,0)|\leq p(x,y)$, which means that

$\left||x-0|^{\alpha}-|y-0|^{\alpha}\right|=\left||x|^{\alpha}-|y|^{\alpha}\right|\leq|x-y|^{\alpha}$. Furthermore, if we assume that $x,y>0$ this yields the desired result.

8. (i) Clearly $0\leq d(x,y)$. Notice that

$\sum_{n=1}^{\infty}\frac{1}{n!}\frac{|x_n-y_n|}{1+|x_n-y_n|}<\sum_{n=1}^{\infty}\frac{1}{n!}=e$ for every $x=\{x_n\}$ and $y=\{y_n\}$. Hence $d(x,y)<\infty$

(ii) $d(x,y)=0$ if and only if $\frac{|x_n-y_n|}{1+|x_n-y_n|}=0$

and since $\frac{|\cdot|}{1+|\cdot|}$ is a distance function on $\mathbb{R}$, it follows that $x_n=y_n$ for all $n\in\mathbb{N}$. Thus $x=y$.

(iii) Clearly $d(x,y) = d(y,x)$.

(iv) Let $z \in \mathbb{R}^\infty$ with $z = \{z_n\}$. Since the function $\frac{|\cdot|}{1+|\cdot|}$ has the triangle inequality property, we see that

$$\frac{|x_n - y_n|}{1+|x_n-y_n|} \leq \frac{|x_n - z_n|}{1+|x_n-z_n|} + \frac{|y_n - z_n|}{1+|y_n-z_n|}.$$ Hence

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n-y_n|}{1+|x_n-y_n|} \leq \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n-z_n|}{1+|x_n-z_n|} + $$

$$+ \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|y_n-z_n|}{1+|y_n-z_n|} = d(x,z) + d(y,z)$$

9. It is clear that d, ρ, and σ satisfy properties (i) and (iii). To see that d satisfies property (ii) observe that if $d(f,g) = 0$ then $|f(t)-g(t)| \leq \max_{t\in[a,b]} |f(t)-g(t)| = 0$, implying that $f(t) = g(t)$ for all $t \in [a,b]$ and therefore that $f = g$. To prove property (ii) for ρ and σ, observe that $h(t) = |f(t)-g(t)|$ and $k(t) = \min\{|f(t)-g(t)|, 1\} = \frac{h(t)+1-|h(t)-1|}{2}$ are nonnegative continuous functions over $[a,b]$. It will suffice, for now, to say that the area under a nonnegative continuous function is 0 if and only if that function is identically 0. Later in the course we will justify this claim more rigorously.

To establish property (iv) for d, ρ, and σ, note that

$$\max_{t\in[a,b]} |f(t)-g(t)| \le \max_{t\in[a,b]} \{|f(t)-s(t)|+|s(t)-g(t)|\} \le$$

$$\le \max_{t\in[a,b]} |f(t)-s(t)| + \max_{t\in[a,b]} |s(t)-g(t)|.$$

Thus $d(f,g) \le d(f,s)+d(s,g)$ where f, g, s are any three elements of $C[a,b]$.

To show that property (iv) holds for ρ and σ, observe that if φ and $\psi \in C[a,b]$ and $\varphi(t) \le \psi(t)$ then $\int_a^b \varphi(t)dt \le \int_a^b \psi(t)\,dt$.

For any two functions $f, g \in C[a,b]$, $|f(t)-g(t)|$ and $\min\{|f(t)-g(t)|, 1\}$ are in $C[a,b]$. Furthermore, $|f(t)-g(t)| \le |f(t)-s(t)| + |s(t)-g(t)|$. Setting $\varphi(t)=|f(t)-g(t)|$ and $\psi(t)=|f(t)-s(t)| + |s(t)-g(t)|$ we get that $\rho(f,g)=\int_a^b \varphi(t)dt \le \int_a^b \psi(t)dt = \int_a^b |f(t)-s(t)|dt + \int_a^b |s(t)-g(t)|dt = \rho(f,s)+\rho(s,g)$. Thus ρ has the triangle inequality property.

Recall that $\min\{|f(t)-g(t)|, 1\} \le \min\{|f(t)-s(t)|, 1\} + \min\{|s(t)-g(t)|, 1\}$. Thus, to show that σ has property (iv) we simply repeat the argument used for ρ.

10. Let $A_1, \ldots, A_n$ be bounded sets in (M, d). Then, for each i, there is some $x_i \in M$ and $r_i \in (0,\infty)$ such that $A_i \subset B_{r_i}(x_i)$. Let $r = \sum_{i=2}^{n} d(x_i, x_1) + \max_{1 \le j \le n} \{r_j\}$.

Then for any $y \in A = \bigcup_{i=1}^{n} A_i$, $y \in A_j$ for some $j \in \{1, \ldots, n\}$ and $d(y, x_1) \le d(y, x_j) + d(x_j, x_1) < r_j + \sum_{i=2}^{n} d(x_i, x_1) \le r$.
Thus, $A \subset B_r(x_1)$, which means that A is bounded.
We have shown that a finite union of bounded sets is bounded.

11. Suppose $A \subset B_r(x)$ for some $x \in M$ and $r \in (0,\infty)$.
This means that for any $a, b \in A$, $d(a,b) \le d(a,x) + d(b,x) < r + r = 2r$. This means that $\sup_{a,b \in A} \{d(a,b)\} \le 2r$. (Why?)

On the other hand, if $\sup_{a,b \in A} \{d(a,b)\} < r$ for some $r \in (0,\infty)$, then for some fixed point $x \in A$, $d(x,b) < r$ for all $b \in A$.
That is, $A \subset B_r(x)$, which implies that A is bounded.

12. Let x be any vector in $\mathbb{R}^n$. Then $x = (x_1, \ldots, x_n)$ and $\|x\|_\infty = \max_{1 \le i \le n} |x_i| = \sqrt{(\max_{1 \le i \le n} |x_i|)^2} \le \sqrt{\sum_{i=1}^{n} |x_i|^2} = \|x\|_2$.
To see that $\|x\|_2 \le \|x\|_1$, observe that $x = \sum_{i=1}^{n} x_i e_i$,

where $e_i = (0,\ldots,1,\ldots,0)$ is the standard basis vector with 1 in the i^{th} component and 0 everywhere else.

By triangle inequality, $\|x\|_2 = \|\sum_{i=1}^{n} x_i e_i\|_2 \le |x_1|\|e_1\|_2 + \ldots + |x_n|\|e_n\|_2$ $= |x_1| + \ldots + |x_n| = \|x\|_1$. Thus $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$ as desired.

To show that $\|x\|_1 \le n\|x\|_\infty$, observe that

$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n| \le \max_{1\le i\le n} |x_i| + \max_{1\le i\le n} |x_i| + \ldots + \max_{1\le i\le n} |x_i|$$

$$= n \max_{1\le i\le n} |x_i| = n\|x\|_\infty.$$

Finally, observe that $\|x\|_1 = (1,1,\ldots,1)\cdot(|x_1|,|x_2|,\ldots,|x_n|)$.

By Cauchy-Schwarz inequality, $\|x\|_1 \le \|(1,\ldots,1)\|_2 \|(|x_1|,\ldots,|x_n|)\|_2$ $= \sqrt{1^2+\ldots+1^2}\sqrt{|x_1|^2+\ldots+|x_n|^2} = \sqrt{n}\,\|x\|_2$

13. If $a,b \in B_r(x)$, then $d(a,b) \le d(a,x) + d(b,x) < r + r = 2r$.
Thus $\sup_{a,b\in B_r(x)} d(a,b) \le 2r$, which means that $\operatorname{diam} B_r(x) \le 2r$.

To see that $\operatorname{diam} B_r(x) < 2r$ can happen, suppose d is discrete. Then $B_1(x) = \{x\}$ and $\operatorname{diam} B_1(x) = 0 < 2\cdot 1$.

14. Suppose $\operatorname{diam} A < r$. Let $a \in A$ be any element in A.
Then, for any $b \in A$, $d(a,b) \le \sup_{a,b\in A} d(a,b) = \operatorname{diam} A < r$.

Therefore $B_1(a) \supset A$.

15. Suppose $A \subset B$. Let $S_A = \{d(x,y) : x, y \in A\}$ and $S_B = \{d(x,y) : x, y \in B\}$. Then $S_A \subset S_B \subset [0,\infty)$. Therefore, if α is an upper bound of S_B, then α must also be an upper bound of S_A. This means that $\operatorname{diam} A = \sup S_A \leq$ $\leq \sup S_B = \operatorname{diam} B$.

16. Let $A = \{1\}$ and $B = \{3\}$. Then $\operatorname{diam} A = \operatorname{diam} B = 0$, while $\operatorname{diam}(A \cup B) = |1-3| = 2$. Thus, $\operatorname{diam}(A \cup B) >$ $> \operatorname{diam} A + \operatorname{diam} B$.

Suppose that $A, B \subset M$ for some metric space (M,d). If $A \cap B \neq \emptyset$. Let $x \in A \cap B$. Then, for any $a, b \in A \cup B$, we have $d(a,b) \leq d(a,x) + d(b,x) \leq \sup\limits_{a,t \in A} d(a,t) + \sup\limits_{b,t \in B} d(b,t) =$ $= \operatorname{diam} A + \operatorname{diam} B$. This actually implies the stronger statement

$$\operatorname{diam}(A \cup B) = \sup_{a,b \in A \cup B} d(a,b) \leq \operatorname{diam} A + \operatorname{diam} B.$$